On boundedness of multidimensional Hausdorff operator in weighted Lebesgue spaces

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Abstract

In this paper the boundedness of multidimensional Hausdorff operator in weighted Lebesgue spaces is proved. In particular, necessary and sufficient condition for the boundedness of multidimensional Hausdorff operator are established in weighted Lebesgue spaces.

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1 Introduction

The investigation of Hausdorff operator can be traced back to 1917 by Hurwith and Silverman in [7] with summability of number series. Therefore Hausdorff operator have become an essential part of modern harmonic analysis. In particular, the study of Hausdorff operator has attracted resurgent attentions in recent years. We refer to [2], [4]-[6], [8]-[11] and [12] for some recent work in this vein.

Let $\mathbb{R}^n$ n-dimensional Euclidean space of points $x=(x_1,x_2,\ldots,x_n)$ and let $\mathcal{A}:=(a_{ij})$ be an $n\times n$ matrix whose entries $a_{ij}:\mathbb{R}^n\to\mathbb{R}$ are Lebesgue measurable functions, $i,j=1,\ldots,n$.. Suppose $f:\mathbb{R}^n\to\mathbb{R}$ is a Lebesgue measurable function. For a fixed kernel function $\varphi\in L_1^{loc}(\mathbb{R}^n)$, the n-dimensional Hausdorff operator is defined in the integral form by

$$\mathcal{H}_\varphi(f)(x)=\int_{\mathbb{R}^n}\varphi(y)\,f\left(x\mathcal{A}(y)\right)\,dy.$$

For the generalized version, we refer to survey papers [3] and [11].

Let $\mathcal{A}:=(a_{ij})$ is a non-singular matrix, so it is invertible. The corresponding multidimensional adjoint Hausdorff operator is defined as follows (see [8])

$$\mathcal{H}^\star_\varphi(f)(x)=\int_{\mathbb{R}^n}\varphi(y)|detA^{-1}(y)|\,f\left(x\mathcal{A}^{-1}(y)\right)\,dy.$$

This integral operator is deeply rooted in the study of one-dimensional Fourier analysis. Particularly, it is closely related to the summability of the classical Fourier series (see [11]). Many important operators of harmonic analysis are special cases of the Hausdorff operator, by taking suitable choice of φ. For example, the Hardy operator, the adjoint Hardy operator, the Cesàro operator, the Hardy-Littlewood-Pólya operator, the Riemann-Liouville fractional derivatives and others can be derived from the Hausdorff operator. The Hausdorff operator has received extensive study in recent years, particularly its boundedness on the Lebesgue space L_p and the Hardy space H_p (see [8]-[11]). Recently, in [1] the boundedness of one-dimensional Hausdorff operator in different Lebesgue type function spaces was proved. Also, in [5] two-weighted inequalities for Hausdorff operators in Herz-type Hardy spaces was proved.

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In this paper the boundedness of multidimensional Hausdorff operator in weighted Lebesgue spaces is proved. In particular, some criterion on function φ is given for the boundedness of multidimensional Hausdorff operator in weighted Lebesgue spaces for power type weight function.

2 Preliminaries

Let $x=(x_1,\ldots,x_n)$ and let $\mathbb{R}^n_{++}=\{x\in\mathbb{R}^n : x_i>0, i=1,\ldots,n\}$. Assume that $\mathbb{R}^1_{++}=\mathbb{R}_+$. Let ω is a weight function, i.e. $\omega(x)>0$ almost everywhere and $\omega\in L_1^{loc}(\mathbb{R}^n)$. Suppose A be a Lebesgue measurable set of $\mathbb{R}^n$. By χ_A we denote the characteristic function of a set A. By $L_{p,\omega}(\mathbb{R}^n)$ we denote the set of Lebesgue measurable functions f that satisfy

$$\|f\|_{L_{p,\omega}(\mathbb{R}^n)}=\left(\int_{\mathbb{R}^n}|f(x)|^p\,\omega(x)\,dx\right)^{\frac{1}{p}}<\infty.$$

3 Main results

Now we reduce the following Theorems.

Theorem 3.1. Let $1<p<\infty$, v and w be weight functions defined on $\mathbb{R}^n$ and let $\left|\det\mathcal{A}^{-1}(y)\right|\neq 0$ for almost every $y\in\mathbb{R}^n$. Suppose

$$B_{\sup}=\int_{\mathbb{R}^n}|\varphi(y)|\left|\det\mathcal{A}^{-1}(y)\right|^{\frac{1}{p}}\left(\sup_{x\in\mathbb{R}^n}\frac{w(x)}{v\left(x\mathcal{A}(y)\right)}\right)^{\frac{1}{p}}dy<\infty.$$

Then for any $f\in L_{p,v}(\mathbb{R}^n)$ the inequality

$$\|\mathcal{H}_\varphi f\|_{L_{p,w}(\mathbb{R}^n)}\le B_{\sup}\|f\|_{L_{p,v}(\mathbb{R}^n)}.$$

holds.

Proof. Applying Minkowski inequality and after change of variables $(x\mathcal{A}(y)=z)$, we have

$$\|\mathcal{H}_\varphi f\|_{L_{p,w}(\mathbb{R}^n)}=\left(\int_{\mathbb{R}^n}w(x)\left|\int_{\mathbb{R}^n}\varphi(y)f\left(x\mathcal{A}(y)\right)dy\right|^p dx\right)^{\frac{1}{p}}$$

$$=\left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\varphi(y)f\left(x\mathcal{A}(y)\right)w^{\frac{1}{p}}(x)dy\right|^p dx\right)^{\frac{1}{p}}$$

$$\le\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\varphi(y)|^p\left|f\left(x\mathcal{A}(y)\right)\right|^p w(x)dx\right)^{\frac{1}{p}}dy$$

$$=\int_{\mathbb{R}^n}|\varphi(y)|\left(\int_{\mathbb{R}^n}\left|f\left(x\mathcal{A}(y)\right)\right|^p w(x)dx\right)^{\frac{1}{p}}dy$$

$$= \int\limits_{\mathbb{R}^n} |\varphi(y)| \left|\det \mathcal{A}^{-1}(y)\right|^{\frac{1}{p}} \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \, w(x\mathcal{A}^{-1}(y)) \, dx \right)^{\frac{1}{p}} dy$$

$$= \int\limits_{\mathbb{R}^n} |\varphi(y)| \left|\det \mathcal{A}^{-1}(y)\right|^{\frac{1}{p}} \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \, v(x) \frac{w\left(x\mathcal{A}^{-1}(y)\right)}{v(x)} \, dx \right)^{\frac{1}{p}} dy$$

$$\leq \left(\int\limits_{\mathbb{R}^n} |\varphi(y)| \left|\det \mathcal{A}^{-1}(y)\right|^{\frac{1}{p}} \left(\sup_{x\in\mathbb{R}^n} \frac{w\left(x\mathcal{A}^{-1}(y)\right)}{v(x)} \right)^{\frac{1}{p}} dy \right) \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \, v(x) \, dx \right)^{\frac{1}{p}}$$

$$= \left(\int\limits_{\mathbb{R}^n} |\varphi(y)| \left|\det \mathcal{A}^{-1}(y)\right|^{\frac{1}{p}} \left(\sup_{x\in\mathbb{R}^n} \frac{w(x)}{v\left(x\mathcal{A}(y)\right)} \right)^{\frac{1}{p}} dy \right) \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \, v(x) \, dx \right)^{\frac{1}{p}}$$

$$= B_{\sup} \|f\|_{L_{p,v}(\mathbb{R}^n)}.$$

Thus,

$$\|\mathcal{H}_\varphi f\|_{L_{p,w}(\mathbb{R}^n)} \leq B_{\sup} \|f\|_{L_{p,v}(\mathbb{R}^n)}.$$

Theorem 3.1 is proved.

In order to investigate the necessity of condition 1), we take the matrix $\mathcal{A}$ as $\mathcal{A}(y)=diag(y_1,\ldots,y_n)$. Then the following Theorem holds.

Theorem 3.2. Let $1 < p < \infty$, v and w be weight functions defined on $\mathbb{R}^n$, $\varphi : \mathbb{R}^n \to \mathbb{R}_+$, and let $\varphi \in L_1^{loc}\left(\mathbb{R}^n\right)$. Suppose

$$B_{\inf} = \int\limits_{\mathbb{R}^n} \varphi(y) \prod_{i=1}^n |y_i|^{-\frac{1}{p}} \inf_{x\in\mathbb{R}^n} \left(\frac{w(x)}{v\left(x\mathcal{A}(y)\right)} \right)^{\frac{1}{p}} dy > 0.$$

If $\mathcal{H}_\varphi : L_{p,w}\left(\mathbb{R}^n\right) \to L_{p,v}\left(\mathbb{R}^n\right)$ is bounded, then

$$\|\mathcal{H}_\varphi\|_{L_{p,w}(\mathbb{R}^n)\to L_{p,v}(\mathbb{R}^n)} \geq B_{\inf}.$$

Proof. It is obvious that if $\mathcal{A}(y) = diag(y_1, \ldots, y_n)$, then

$$\mathcal{H}_\varphi(f)(x) = \int_{\mathbb{R}^n} \varphi(y) \, f\left(x_1 y_1, \ldots, x_n y_n\right) \, dy.$$

We denote $A_i := \{x_i : x_i \in \mathbb{R}, \, |x_i| \geq 1\}\,,\ i = 1, \ldots, n$. Let us fix $0 < \varepsilon < 1$ and define the function

$$f_\varepsilon(x) = \prod_{i=1}^n |x_i|^{-\frac{1}{p}-\varepsilon} \chi_{A_i}(x) \, v(x)^{-\frac{1}{p}}.$$

Then, by straightforward calculations we get

$$\|f_\varepsilon\|_{L_{p,v}(\mathbb{R}^n)} = \left(\frac{2}{\varepsilon p}\right)^{\frac{n}{p}}.$$

On the other hand, we have

$$\mathcal{H}_\varphi f_\varepsilon(x) = \prod_{i=1}^n |x_i|^{-\frac{1}{p}-\varepsilon} \int_C \varphi(y) \prod_{i=1}^n |y_i|^{-\frac{1}{p}-\varepsilon} v\,(x\mathcal{A}(y))^{-\frac{1}{p}}\, dy,$$

where $C = \left\{y : y = (y_1, \ldots, y_n)\,, |y_i| \ge \frac{1}{|x_i|},\ i = 1, \ldots, n\right\}$. Thus,

$$\|\mathcal{H}_\varphi f_\varepsilon\|_{L_{p,w}(\mathbb{R}^n)}$$

$$= \left(\int_{\mathbb{R}^n} \prod_{i=1}^n |x_i|^{-1-p\varepsilon} \left(\int_C \varphi(y) \prod_{i=1}^n |y_i|^{-\frac{1}{p}-\varepsilon} \left(\frac{w(x)}{v\,(x\mathcal{A}(y))}\right)^{\frac{1}{p}} dy\right)^p dx\right)^{\frac{1}{p}}$$

$$\ge \left(\prod_{i=1}^n \int_{A_i} |x_i|^{-1-p\varepsilon} \left(\int_C \varphi(y) \prod_{i=1}^n |y_i|^{-\frac{1}{p}-\varepsilon} \left(\frac{w(x)}{v\,(x\mathcal{A}(y))}\right)^{\frac{1}{p}} dy\right)^p dx\right)^{\frac{1}{p}}$$

$$\ge \left(\prod_{i=1}^n \int_{A_{i,\varepsilon}} |x_i|^{-1-p\varepsilon} \left(\int_{C_\varepsilon} \varphi(y) \prod_{i=1}^n |y_i|^{-\frac{1}{p}-\varepsilon} \left(\frac{w(x)}{v\,(x\mathcal{A}(y))}\right)^{\frac{1}{p}} dy\right)^p dx\right)^{\frac{1}{p}}$$

$$\ge B(\varepsilon) \left(\prod_{i=1}^n \int_{A_{i,\varepsilon}} |x_i|^{-1-p\varepsilon}\, dx\right)^{\frac{1}{p}}$$

$$= B(\varepsilon) \left(\frac{2}{\varepsilon p}\right)^{\frac{n}{p}} \varepsilon^{n\varepsilon} = \|f_\varepsilon\|_{L_{p,v}(\mathbb{R}^n)} B(\varepsilon)\, \varepsilon^{n\varepsilon}, \tag{1}$$

where $A_{i,\varepsilon} := \left\{x_i : x_i \in \mathbb{R},\, |x_i| \ge \frac{1}{\varepsilon}\right\}$, $C_\varepsilon := \{y : y = (y_1, \ldots, y_n)\,, |y_i| \ge \varepsilon\}$, $i = 1, \ldots, n$ and

$$B(\varepsilon) = \int\limits_{C_\varepsilon} \varphi(y) \prod_{i=1}^n |y_i|^{-\frac{1}{p}-\varepsilon} \inf_{x\in\mathbb{R}^n} \left(\frac{w(x)}{v\,(x\mathcal{A}(y))}\right)^{\frac{1}{p}} dy.$$

Next, by (1), we get

$$\|\mathcal{H}_\varphi\|_{L_{p,w}(\mathbb{R}^n)\to L_{p,v}(\mathbb{R}^n)} \ge \varepsilon^{n\varepsilon} B(\varepsilon).$$

Finally, by virtue of the Fatou lemma we pass to the limit $\varepsilon \to 0$ and we get

$$\|\mathcal{H}_\varphi\|_{L_{p,w}(\mathbb{R}^n)\to L_{p,v}(\mathbb{R}^n)} \ge \int\limits_{\mathbb{R}^n} \varphi(y) \prod_{i=1}^n |y_i|^{-\frac{1}{p}} \inf_{x\in\mathbb{R}^n} \left(\frac{w(x)}{v\,(x\mathcal{A}(y))}\right)^{\frac{1}{p}} dy,$$

and this ends the proof of the Theorem 3.2.

It is obvious that if $\mathcal{A}(y) = diag(y_1, \ldots, y_n)$, then

$$\left|\det \mathcal{A}^{-1}(y)\right| = \frac{1}{\prod\limits_{i=1}^n |y_i|}.$$

In the case of a diagonal matrix $\mathcal{A}(y) = diag(y_1, \ldots, y_n)$ following important corollaries are hold.

Corollary 3.3. Let $1 < p < \infty$ and v, w be weight functions defined on $\mathbb{R}^n$ such that

$$\sup_{x\in\mathbb{R}^n} \frac{w(x)}{v\left(x\mathcal{A}(y)\right)} \le C \inf_{x\in\mathbb{R}^n} \frac{w(x)}{v\left(x\mathcal{A}(y)\right)}$$

for some constant $C \ge 1$ independent of $y \in \mathbb{R}^n$. Then $\mathcal{H}_\varphi : L_{p,w}\left(\mathbb{R}^n\right) \to L_{p,v}\left(\mathbb{R}^n\right)$ is bounded if and only if $B_{\sup} < \infty$. Furthermore,

$$\frac{1}{C} B_{\sup} \le \|H_\varphi\|_{L_{p,w}(\mathbb{R}^n)\to L_{p,v}(\mathbb{R}^n)} \le B_{\sup}.$$

Corollary 3.4. Let $1 < p < \infty$ and $v(x) = w(x) = \prod\limits_{i=1}^{n} |x_i|^{\alpha_i}$ and let $\alpha_i > -1$, $i = 1, \ldots, n$. Then

$$\|H_\varphi\|_{L_{p,w}(\mathbb{R}^n)\to L_{p,v}(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^n} \varphi(y) \prod_{i=1}^{n} |y_i|^{-\frac{\alpha_i+1}{p}}\, dy.$$

Corollary 3.5. Let $1 < p < \infty$, and let $f : \mathbb{R}^n_{++} \to \mathbb{R}$ be a Lebesgue measurable function and $\varphi(y) = \chi_{[0,1]^n}(y)$. Suppose $v(x) = w(x) = \prod\limits_{i=1}^{n} x_i^{\alpha_i}$ and $-1 < \alpha_i < p-1$, $i = 1, \ldots, n$. Then

$$\mathcal{H}_\varphi f(x) = \mathcal{C} f(x) = \int\limits_0^1 \cdots \int\limits_0^1 f\left(x_1 y_1, \ldots, x_n y_n\right) dy_1 \cdots dy_n$$

$$= \frac{1}{x_1 \cdots x_n} \int\limits_0^{x_1} \cdots \int\limits_0^{x_n} f\left(t_1, \ldots, t_n\right) dt_1 \cdots dt_n$$

and

$$\|\mathcal{C}\|_{L_{p,w}(\mathbb{R}^n)\to L_{p,v}(\mathbb{R}^n)} = \frac{p^n}{\prod\limits_{i=1}^{n} \left(p - 1 - \alpha_i\right)}.$$

Corollary 3.6. Let $1 < p < \infty$, and let $f : \mathbb{R}^n_{++} \to \mathbb{R}$ be a Lebesgue measurable function and $\varphi(y) = \dfrac{\chi_{[0,1]^n}(y)}{y_1 \cdots y_n}$, and $\mathcal{A}\left(y_1, \ldots, y_n\right) = diag\left(\frac{1}{y_1}, \ldots, \frac{1}{y_n}\right)$. Suppose $v(x) = w(x) = \prod\limits_{i=1}^{n} x_i^{\alpha_i}$ and $\alpha_i > -1$, $i = 1, \ldots, n$. Then

$$\mathcal{H}_\varphi f(x) = \mathcal{C}^\star f(x) = \int\limits_0^1 \cdots \int\limits_0^1 \frac{1}{y_1 \cdots y_n} f\left(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n}\right) dy_1 \cdots dy_n$$

$$= \int\limits_{x_1}^{\infty} \cdots \int\limits_{x_n}^{\infty} \frac{f\left(t_1, \ldots, t_n\right)}{t_1 \cdots t_n} dt_1 \cdots dt_n$$

and

$$\|\mathcal{C}^\star\|_{L_{p,w}(\mathbb{R}^n)\to L_{p,v}(\mathbb{R}^n)} = \frac{p^n}{\prod\limits_{i=1}^{n} \left(\alpha_i + 1\right)}.$$

Remark 3.7. Note that for one-dimensional Hausdorff operator Theorem 3.1 and Theorem 3.2 was proved in [1]. Also, the boundedness of multidimensional Hausdorff operator in different function spaces was proved in [3]-[7], [9]-[12] and others.

The same results hold for multidimensional adjoint Hausdorff operator.

Theorem 3.8. Let $1 < p < \infty$, v and w be weight functions defined on $\mathbb{R}^n$ and let $|\det \mathcal{A}(y)| \neq 0$ for almost every $y \in \mathbb{R}^n$. Suppose

$$B^{\star}_{\sup} = \int\limits_{\mathbb{R}^n} |\varphi(y)| \, |\det \mathcal{A}(y)|^{\frac{1}{p}-1} \left(\sup_{x\in\mathbb{R}^n} \frac{w\left(x\mathcal{A}(y)\right)}{v(x)} \right)^{\frac{1}{p}} dy < \infty.$$

Then for any $f \in L_{p,v}\left(\mathbb{R}^n\right)$ the inequality

$$\left\| \mathcal{H}^{\star}_{\varphi} f \right\|_{L_{p,w}(\mathbb{R}^n)} \leq B^{\star}_{\sup} \|f\|_{L_{p,v}(\mathbb{R}^n)}.$$

holds.

Let $\mathcal{A}$ is a diagonal matrix, i.e. $\mathcal{A}(y) = diag(y_1, \ldots, y_n)$. Then the following Theorem is holds.

Theorem 3.9. Let $1 < p < \infty$, v and w be weight functions defined on $\mathbb{R}^n$, $\varphi : \mathbb{R}^n \to \mathbb{R}_+$, and let $\varphi \in L_1^{loc}\left(\mathbb{R}^n\right)$. Suppose

$$B^{\star}_{\inf} = \int\limits_{\mathbb{R}^n} \varphi(y) \prod_{i=1}^{n} |y_i|^{\frac{1}{p}-1} \inf_{x\in\mathbb{R}^n} \left(\frac{w(x\mathcal{A}(y))}{v(x)} \right)^{\frac{1}{p}} dy > 0.$$

If $\mathcal{H}^{\star}_{\varphi} : L_{p,w}\left(\mathbb{R}^n\right) \to L_{p,v}\left(\mathbb{R}^n\right)$ is bounded, then

$$\|\mathcal{H}^{\star}_{\varphi}\|_{L_{p,w}(\mathbb{R}^n)\to L_{p,v}(\mathbb{R}^n)} \geq B^{\star}_{\inf}.$$

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